STEINITZ' THEOREM FOR POLYHEDRA

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Abstract. We prove Steinitz' theorem for polyhedra, stating that G is the graph of a polyhedron if and only if G is simple, planar, and 3-connected.

1. Introduction

An *undirected graph* is a set of vertices and edges, where each edge connects two vertices. From any polyhedron one can form a graph, called the *graph of the polyhedron*, by letting the vertices of the graph correspond to vertices of the polyhedron and by joining two vertices of the graph whenever the corresponding vertices on the polyhedron are the endpoints of an edge of the polyhedron. The graph K_4 on four vertices such that every pair of vertices is connected is the graph of a tetrahedron, see Figure 1.

A graph *simple* if every edge is between two distinct vertices and there is no pair edges that connect the same pair of vertices. A graph is *planar* if we can represent with vertices as points on the plane and edges as curves with vertex endpoints such that no two edge curves cross on their interiors. A graph is 3-*connected* if whenever we remove one or two of its vertices and the edges incident to those vertices, the graph remains connected.

Steinitz' theorem states that planarity and 3-connectedness are necessary and sufficient conditions to characterize the graphs of convex polyhedra.

Theorem 1.1 (Steinitz' Theorem [SR76]). *A graph G is the graph of a polyhedron if and only if G is simple, planar, and* 3*-connected.*

We can see that K_4, the graph of a tetrahedron, is 3-connected: any time we remove one or two vertices and the edges incident to them, we obtain a connected graph. We can also see that the graph of the dodecahedron is 3-connected. See Figure 2 for an example.

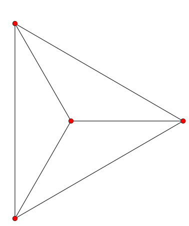

Figure 1.

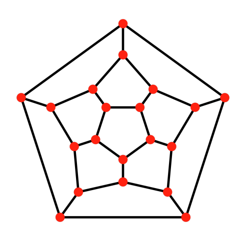

Figure 2.

Conversely, if we are given a 3-connected graph, Steinitz' Theorem allows us to make it into the graph of a polytope, see Figure 3.

Remark. No similar theorem that characterizes graphs of higher dimensional polyhedra, known as *polytopes*, is known. See Chapter 4 of [Zie95] for further discussion.

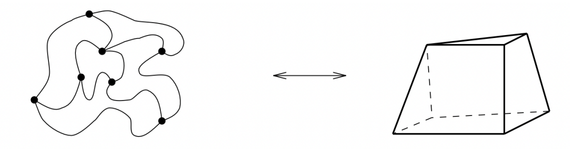

FIGURE 3.

Acknowledgments. I would like to thank my DRP mentor Sameera Vemulapalli for choosing such an interesting topic and for helping me all throughout the process.

Outline. For the necessary graph theory tools, read section 2. To show every simple 3-connected graph is the graph of a polytope, read sections 3 and 5. To show the graph of a polytope is simple and 3-connected, read sections 4 and 5.

2. Graph theory preliminaries

Let's admit graphs having loops and multiple edges between pairs of vertices. Denote $V(G)$ and $E(G)$ to be the vertices and edges in G. We also sometimes denote a graph $G = (V, E)$, where V and E are the vertices and edges of G.

Definition 2.1. In a graph G, the *degree* of a vertex $v \in V(G)$ is the number of edges incident to v.

Definition 2.2. A *path* is a non-empty graph $P = (V, E)$ of the form

$$V = \{x_0, x_1, \ldots, x_k\}, \qquad E = \{x_0x_1, x_1x_2, \ldots, x_{k-1}x_k\},$$

where the x_i are all distinct.

Definition 2.3. A graph G is *connected* if it is non-empty and any two of its vertices are linked by a path in G.

Definition 2.4. In a graph G, a set of vertices $X \subseteq V(G)$ is called a *separator* if there exist vertices $a, b \notin X$ such that any path from a to b passes through a vertex of X.

Definition 2.5. For a positive integer k, a graph G is *k-connected* if $|V(G)| > k$ and there does not exist a separator with fewer than k vertices.

From this point onwards, we assume that a 2-connected graph has no loops and that a 3-connected graph is simple.

Note that 1-connected is equivalent to connected. Below on the left is an example of a graph H that is 2-connected, but not 3-connected. The graph G on the right is called the *octahedron* graph. It is 4-connected.

We introduce two basic operations on graphs.

Definition 2.6. Let $G = (V, E)$ be a graph. A *deletion* of an edge $e \in E$ gives another graph $G' = (V, E \setminus \{e\})$. A *contraction* of an edge $uv \in E$ creates another graph for which the two vertices of the edge are identified, i.e. a new graph G'' with vertices $(V \setminus \{u, v\}) \cup \{w\}$ such that all edges between vertices in $V \setminus \{u, v\}$ are preserved and w is connected to any vertex $x \in V \setminus \{u, v\}$ if and only if $ux \in E$ or $vx \in E$.

Definition 2.7. A *minor* of G is a graph that can be obtained from G via a sequence of deletions and contractions of edges.

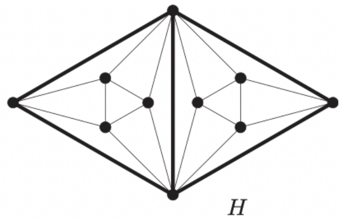

FIGURE 4. From [Die17]

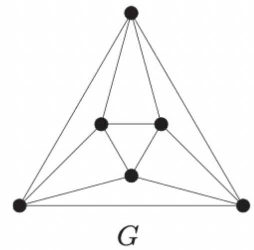

FIGURE 5. From [Die17]

3. Delta-Wye operations on graphs

In this section we explain how to build 3-connected planar graphs from K_4 via an operation that preserves realizability, i.e. the operation turns a graph of a polytope into a graph of another polytope. This operation is called the *Delta-Wye operation*.

Definition 3.1. An *in series* contraction, shown in Figure 6, is an edge contraction of an edge incident to a vertex of degree 2. An *in parallel* contraction, shown in Figure 7, is an edge contraction of an edge parallel to another edge.

Any sequence of in series contractions and in parallel deletions is called an *series-parallel reduction*, or *SP-reduction*.

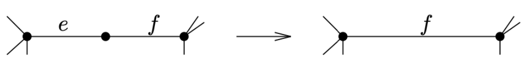

FIGURE 6. From [Zie95]

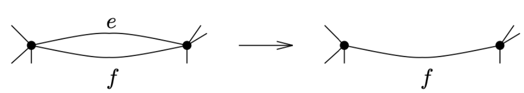

FIGURE 7. From [Zie95]

Definition 3.2. A *Delta-Wye operation*, or *ΔY operation*, replaces a triangle that bounds a face by a 3-star that connects the same vertices, or vice versa, as shown in Figure 8. If we want to specify the direction of the transformation, then we will call it a *Δ-to-Y transformation*, respectively a *Y-to-Δ transformation*.

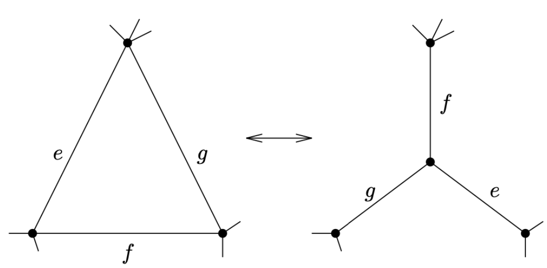

FIGURE 8. From [Zie95]

A ΔY transformation might create series or parallel edges, which can then be SP-reduced.

Lemma 3.3. (1) *Let G be a 2-connected graph, and let $\{e, f, g\}$ be the edges at a vertex v of degree 3 in G.*

If none of its edges are parallel (i.e., if v has three different neighbors), then the result of a Y-to-Δ operation is again 2-connected.

(2) *Let G be a 3-connected graph (in particular, there are no parallel edges; all vertex degrees are at least 3) that is not K_4. Let $\{e, f, g\}$ be the edges at a vertex v in G of degree 3.*

If we perform a Y-to-Δ operation on this 3-star, and then delete all parallel edges created by this (i.e., all edges that originally connected neighbors of v), then the resulting graph is 3-connected.

Proof. Consider a Y-to-Δ operation $G \to G'$, and let w be the degree 3 vertex of the star in G. It is clear that the vertices of G' can be identified with G. If a set X of one or two vertices is separating in G', then there exist two vertices $u, v \in V(G')$ such that any path between them passes through a vertex of X. We claim that any path between u, v in G must pass through a vertex of X. Consider a path P from u to v in G. If P passes through w, then it must contain exactly 2 of the edges e, f, g. By contracting one of the two edges, it becomes isomorphic to a path of G' from u to v. So, X is a separating set for G. It is possible that the Y-to-Δ operation can create parallel edges, but parallel edges cannot belong to P, so to take into account the second part of Lemma 3.3, we delete any remaining parallel edges. This completes the proof.

There is a dual statement of Lemma 3.3, which shows that 2-connectivity and 3-connectivity is preserved after a Δ-to-Y transformation.

Lemma 3.4. (1) *Let G be a 2-connected graph, and let $\{e, f, g\}$ be edges that pairwise overlap at a vertex but do not all overlap.*

Then the result of a Δ-to-Y, and remove all series edges via SP-reductions, operation is again 2-connected.

(2) *Let G be a 3-connected graph and let $\{e, f, g\}$ be edges that pairwise overlap at a vertex but do not all overlap.*

If we perform a Δ-to-Y operation on this triangle, and remove all series edges via SP-reductions, then the resulting graph is 3-connected.

We will omit the proof of the dual statement. The simplest way to approach the proof is to reduce to the statement of Lemma 3.3 via the *dual polyhedron* and *dual graph*. Informally, the *dual polyhedron* of the polyhedron P has vertices that are the faces of P, and two vertices of the dual polyhedron are connected if the corresponding faces of P share an edge. Similarly, the *dual graph* of a planar graph G has vertices that are the regions of G in a planar drawing, and two vertices of the dual graph are connected if the corresponding regions share an edge. See [Zie95] Chapter 4 for more details.

Let C_2 be the graph on two vertices with two parallel edges between them.

Definition 3.5. A 2-connected graph G *ΔY-reducible* if it can be transformed into the graph C_2 by a sequence of ΔY transformations and SP-reductions.

Lemma 3.6. *If a planar graph G is ΔY-reducible, then so is every 2-connected minor H of G.*

Proof. We induct on the number of reductions it takes to reduce G into C_2. If $G = C_2$, the only 2-connected minor of G is itself. Suppose G undergoes one reduction, and let G' be the resulting graph. If the reduction step is a SP-reduction, then H is a minor of G' because H is simple. So, we can apply the inductive hypothesis on G'.

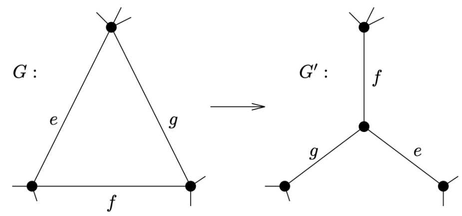

FIGURE 9. A Δ-to-Y reduction step $G \to G'$, from [Zie95]

Now suppose the first reduction is a Δ-to-Y reduction, and let e, f, g be the three edges involved. If all e, f, g are in H, then e, f, g form a non-separating triangle in H, so we can perform a Δ-to-Y step on that triangle in H to obtain a graph H'. Then H' is a minor of G', and we are again finished by induction. Now suppose e, f, g do not all appear in H. If we contracted only one of e, f, g to form H from G, then we have two parallel edges in H, and if we contracted two of e, f, g, then we have a loop in H. If we contract all three of e, f, g, that is the same as deleting one of the edges first, and then contracting the others. So, assume we form the minor H from G by first deleting e. Then H is a minor of G' beacuse we can contract the corresponding edge e' in G', see Figure 9. So, we can apply the inductive hypothesis on G', which completes the proof.

The following class of graphs will be especially important for the proof of Steinitz' theorem.

Definition 3.7. A grid graph $G(m, n)$ is the graph with lattice vertices $\{(a, b) : 0 \le a \le m-1, 0 \le b \le n-1\}$ such that two vertices are connected if and only if their x-coordinates differ by 1 or their y-coordinates differ by 1.

Lemma 3.8. *If G is planar, then it is a minor of a grid graph.*

A sketch of this proof can be found in [Zie95]. We approach this proof analytically.

Proof. Let G be a planar graph with a fixed embedding into $\mathbb{R}^2$. We can 'split' the vertices of G so that G has vertex degree at most 4, as shown in Figure 10.

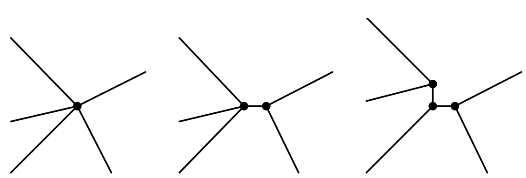

FIGURE 10.

Let $v_1, \ldots, v_n$ be the vertices of G and let $e_1, \ldots, e_m$ denote the edges of G. For $r \in \mathbb{R}$ and $v \in \mathbb{R}^2$, let $B_r(v)$ denote the closed radius r Euclidean ball around v in $\mathbb{R}^2$. Let $\Delta B_r(v)$ denote the boundary of the radius r Euclidean ball around v in $\mathbb{R}^2$.

Let ϵ_1 and ϵ_2 and ϵ_3 be some sufficiently small positive rational numbers such that the following three conditions hold.

(1) Consider the vertex v_j. Choose a rational point $v_j^0 \in B_{\epsilon_3}(v_j)$.
(2) For all i, j, $\#(\Delta B_{\epsilon_1}(v_i^0) \cap e_j) = 1$ if e_j is adjacent to v_i^0 and $\Delta B_{\epsilon_1}(v_i^0) \cap e_j = \emptyset$ otherwise.
(3) The regions

$$\bigcup_{p \in e_i} B_{\epsilon_2}(p) \setminus \bigcup_{j=1}^{n} B_{\epsilon_1}(v_j^0)$$

are connected and disjoint for $1 \le i \le m$.

Consider the vertex v_j. Suppose without loss of generality that the edges $e_1, \ldots, e_k$ for some $k \le 4$ are precisely the edges adjacent to v_j. For $1 \le i \le k$, choose $p_{i;j} \in \Delta B_{\epsilon_1}(v_j^0)$ with rational coordinates such that the distance between p_i and $\Delta B_{\epsilon_1}(v_j^0) \cap e_i$ is strictly less than ϵ_2.

Now, it is clear that by choosing a sufficiently small grid we may draw nonintersecting grid paths from v_j^0 to each of the $p_{i;j}$ for $1 \le i \le k$. In particular, start at 12:00 on the circle $\Delta B_{\epsilon_1}(v_j^0)$. Reorder the edges $e_1, \ldots, e_k$ such that the direction of the edges as one travels clockwise around $\Delta B_{\epsilon_1}(v_j^0)$ is $e_1, \ldots, e_k$.

Draw a path from v_j^0 to $p_{1;j}$ as follows. Start at v_j^0 and travel as follows. If we are one grid step away from $p_{1;j}$, travel to $p_{1;j}$. If it is possible to travel one grid step up without leaving the interior of the disc $B_{\epsilon_1}(v_j^0)$, then travel up one step. Otherwise, if it is possible to travel one grid step right without leaving the interior of the disc $B_{\epsilon_1}(v_j^0)$, then travel right one step. Otherwise, if it is possible to travel one grid step down without leaving the interior of the disc $B_{\epsilon_1}(v_j^0)$, then travel right one step. Otherwise, if it is possible to travel one grid step left without leaving the interior of the disc $B_{\epsilon_1}(v_j^0)$, then travel right one step.

Draw a path from v_j^0 to $p_{2;j}$, if $k \ge 2$ as follows. Start at v_j^0 and travel as follows. Take one step right. If we are one grid step away from $p_{2;j}$, travel to $p_{2;j}$. If it is possible to travel one grid step up without leaving the interior of the disc $B_{\epsilon_1}(v_j^0)$ or intersecting a previous path, then travel up one step. Otherwise, if it is possible to travel one grid step right without leaving the interior of the disc $B_{\epsilon_1}(v_j^0)$ or intersecting a previous path, then travel right one step. Otherwise, if it is possible to travel one grid step down without leaving the interior of the disc $B_{\epsilon_1}(v_j^0)$ or intersecting a previous path, then travel right one step. Otherwise, if it is possible to travel one grid step left without leaving the interior of the disc $B_{\epsilon_1}(v_j^0)$, then travel right one step.

Draw a path from v_j^0 to $p_{3;j}$, if $k \ge 3$, as follows. Start at $v_{j'}$ and travel as follows. Take one step down. If we are one grid step away from $p_{2;j}$, travel to $p_{2;j}$. If it is possible to travel one grid step up without leaving the interior of the disc $B_{\epsilon_1}(v_j^0)$ or intersecting a previous path, then travel up one step. Otherwise, if it is possible to travel one grid step right without leaving the interior of the disc $B_{\epsilon_1}(v_j^0)$ or intersecting a previous path, then travel right one step. Otherwise, if it is possible to travel one grid step down without leaving the interior of the disc $B_{\epsilon_1}(v_j^0)$ or intersecting a previous path, then travel right one step. Otherwise, if it is possible to travel one grid step left without leaving the interior of the disc $B_{\epsilon_1}(v_j^0)$, then travel right one step.

Lemma 3.9. *All grid graphs $G(m,n)$ for $m, n \ge 3$ are ΔY-reducible to K_4.*

Proof. We will reduce $G(m, n)$ to $G(3, 3)$ primarily using two operations. If an edge connects a vertex of degree 3, then we can delete it via a Δ-to-Y transformation and then a series reduction, as shown in Figure 11. If an edge connects two vertices of degree 4, we can "move" the edge to the other side via a Δ-to-Y transformation and then Y-to-Δ transformation, as shown in Figure 12.

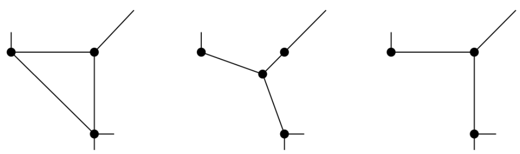

FIGURE 11. Operation 1, from [Zie95]

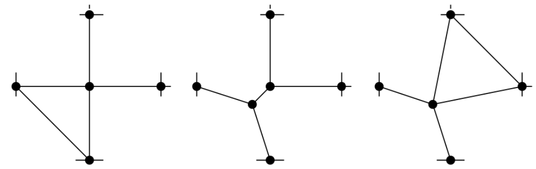

FIGURE 12. Operation 2, from [Zie95]

First we delete the squares in the bottom row. Perform a series reduction on the bottom left and top right corners to get triangles, as shown in the first diagram of Figure 13. Then, using operation 2, move the bottom left edge to the top row. If the obtained edge is parallel to the edge in the top right corner, perform a parallel reduction, see Figure 14. Otherwise, perform operation 1, see the last diagram in Figure 13. This series of steps removes a square in the bottom row. Note that if we are deleting the last square in the row, we can perform two series reductions and a parallel reduction.

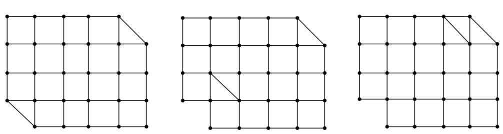

FIGURE 13. From [Zie95]

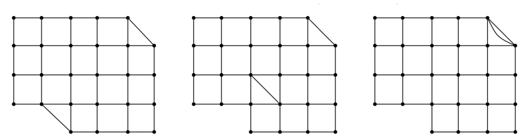

FIGURE 14. From [Zie95]

In this way, we can delete the squares in the bottom row. We can similarly delete the squares in the leftmost column until we obtain a $G(3, 3)$. Once we have a $G(3, 3)$, we reduce to a K_4 via the steps as shown in Figure 15.

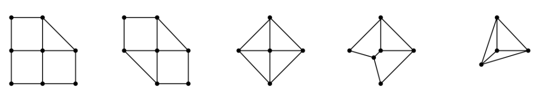

FIGURE 15. From [Zie95]

This completes the proof.

Definition 3.10. A *simple* ΔY *operation* is any ΔY operation followed by all the possible SP-reductions.

Corollary 3.11. *Every* 3*-connected planar graph* G *can be reduced to* K_4 *by a sequence of simple* ΔY *operations.*

Proof. We induct on the number of edges in G. The smallest 3-connected planar graph has 4 vertices, and they all must be connected. This gives us a K_4, which is already reduced.

Now consider any 3-connected planar graph G. By Lemma 3.8, G is a minor of a grid graph $G(m,n)$. We can assume $m, n \geq 3$. By Lemma 3.9, $G(m,n)$ is ΔY reducible to a K_4. Note that $G(m,n)$ is also ΔY-reducible, as we can continue via SP-reductions to reduce $G(m,n)$ to C_2. Since G is a minor of $G(m,n)$, by Lemma 3.6, G is ΔY-reducible. Follow the ΔY reduction of G until parallel or series edges are created. After we follow an SP-reduction, by Lemma 3.3 or Lemma 3.4, the remaining graph G' is 3-connected. Simple ΔY operations also preserve planarity, so G' is planar. A simple ΔY operation decreases the number of edges, so we can apply the inductive hypothesis on G' to obtain that G' can be reduced to K_4 by a sequence of simple ΔY operations. Therefore, G can be reduced to K_4 by a sequence of simple ΔY operations, which completes the induction.

4. The simplex algorithm

Let P be a full-dimensional polyhedron in $\mathbb{R}^3$, and let $V(P)$ be the set of vertices of P. For a point $c = (c_1, c_2, c_3) \in \mathbb{R}^3$, consider the linear function that maps $x = (x_1, x_2, x_3)$ to $c \cdot x = c_1x_1 + c_2x_2 + c_3x_3$.

Theorem 4.1 (Simplex algorithm [Dan90])**.** *The following algorithm finds a vertex of P that maximizes $c \cdot x$.*

(1) *Select a vertex $v \in V(P)$, and suppose it has neighbors $N(v) = \{u_1, \ldots, u_k\}$.*
(2) *If there exists some i such that $c \cdot u_i > c \cdot v$, choose one such i arbitrarily and repeat step 1 with the vertex u_i. If there does not exist such an i, return the vertex v.*

We prove the validity of the algorithm below.

Proof. Note that this algorithm terminates because at each step, the value of $c \cdot x$ increases. It suffices to show that if we are at a vertex v for which $c \cdot v$ is not maximal, we can find a neighbor $u_i \in N(v)$ such that $c \cdot u_i > c \cdot v$. For that we need this claim.

Lemma 4.2. *The cone at v spanned by the neighbors of v contains P:*

$$P \subseteq v + \left\{ x \in \mathbb{R}^3 : x = v + \sum_{i=1}^{k} \lambda_i (u_i - v), \lambda_i \geq 0 \right\}.$$

Proof. The rigorous proof involves many more properties of polyhedra that we can get into, so we refer to [Zie95] Lemma 3.6. A diagram of Claim 4.2 is shown below in Figure 16.

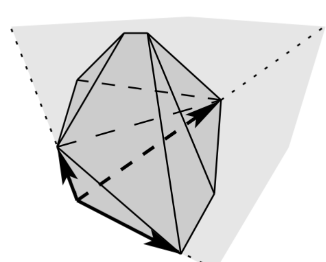

Figure 16. From [Zie95]

Suppose $c \cdot u_i \leq c \cdot v$ holds for all $i = 1, \ldots, k$. Then by Claim 4.2, we can represent a point $x \in P$ by

$$x = v + \lambda_1(u_1 - v) + \cdots + \lambda_k(u_k - v)$$

for some $\lambda_1, \ldots, \lambda_k \geq 0$. Then

$$c \cdot x = c \cdot v + \lambda_1(c \cdot (u_1 - v)) + \cdots + \lambda_k(c \cdot (u_k - v)) \leq c \cdot v,$$

which contradicts v not being a maximal value of $c \cdot x$.

Theorem 4.3 (Balinski [Bal61]). *The graph of a polyhedron is* 3*-connected.*

Proof. Let P be a polyhedron, and let S be a set of two vertices of $V(P)$. Take another vertex $v_0 \in V(P) \setminus S$ and consider a plane $c_1x_2 + c_2x_2 + c_3x_3 = b$ through $S \cup \{v_0\}$ with normal $c = (c_1, c_2, c_3)$. This plane defines a linear function f on $\mathbb{R}^3$:

$$f : x \in \mathbb{R}^3 \mapsto c_1x_1 + c_2x_2 + c_3x_3.$$

Consider the halfspace $H^+ = \{x \in \mathbb{R}^3 : f(x) > b\}$. By the simplex algorithm, we can find a vertex $v_0 \in V(P)$ which maximizes the value of f. Each vertex in H^+ and v_0 can be connected to the maximal point of f via the simplex algorithm. Similarly, by negating the normal of the plane, we can connect every vertex in the halfspace $H^- = \{x \in \mathbb{R}^3 : f(x) < b\}$ and v_0 to the minimum value of f. Since v_0 is connected to all vertices in $V(P) \setminus S$, the graph on $V(P) \setminus S$ is connected. This completes the proof.

See Figure 17 for an example of the proof of Balinski's theorem on a dodecahedron. After choosing the two vertices in S, shown in yellow, to be removed, we choose one of the green vertices to be v_0 and the plane through $S \cup \{v_0\}$. We connect all vertices in $V(P) \setminus S$ to v_0 as in the proof of Balinski's theorem.

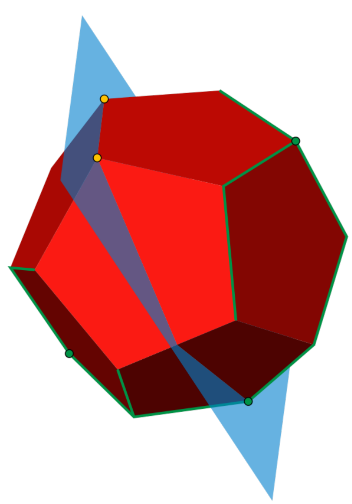

FIGURE 17.

5. Proof of Steinitz' theorem

First we show that Steinitz' theorem is true for 3-connected planar graphs that can be reduced by a simple ΔY transformation.

Lemma 5.1. *Let G be a 3-connected planar graph, and let the graph G^0 be derived from G by a simple ΔY transformation.*

If G^0 is the graph of a 3-polytope, then G is the graph of a 3-polytope.

Proof. Let P^0 be the polytope that has graph G^0. Suppose the ΔY operation is a Δ-to-Y operation. Then corresponding simple ΔY operation corresponds to cutting off vertex at the star of the polyope by a suitable plane.

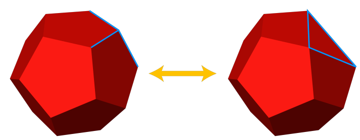

Figure 18.

We will outline a few cases and the others follow with a similar choice of a plane. Consider the simple ΔY transformation from G to G^0 shown on the left of Figure 19. Note that P is created from P^0 by cutting vertex 4^0 from P^0 by a plane through $1^0, 2^0$, and 3^0.

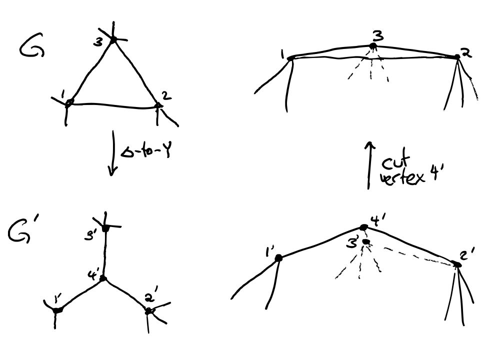

Figure 19. Δ-to-Y example 1

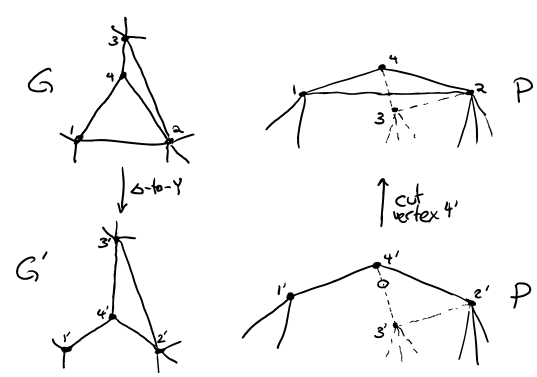

Figure 20. Δ-to-Y example 2

In Figure 20 is another example of the polytope P corresponding to graph G before G undergoes a Δ-to-Y operation. Note that P is created from P^0 by cutting vertex 4^0 from P^0 by a plane through 1^0, 2^0, and the circled point on segment $4^0 3^0$, shown on the right.

Now we suppose the ΔY operation is a Y-to-Δ operation. Consider the simple ΔY transformation from G to G^0 shown in Figure 21. In the diagram, F_1, F_2, F_3 are the faces of the planar graph G^0, which correspond to faces F_1, F_2, F_3 in P^0. We extend F_1 and F_2 past their edges, where they meet at a line ℓ. We pick a suitable point 1 on ℓ such that 1 is not on the plane containing F_3. Then P is the polyhedron formed by the vertices of P^0 with the additional vertex 1.