# DIAGONALIZABLE OPERATORS EXERCISE

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Abstract. One of my favorite linear algebra exercises.

## 1. Introduction

An *operator* on a finite dimensional vector space $V$ over a field $k$ is a linear function $\varphi : V \to V$. An operator is *diagonalizable* if there exists a basis $\mathcal{B}$ of $V$ consisting of eigenvectors: for every $v \in \mathcal{B}$, $\varphi(v) = \lambda v$ for some constant $\lambda$ in the field $k$. A pair of operators $\varphi, \psi$ *commute* if $\varphi \circ \psi = \psi \circ \varphi$. A pair of operators $\varphi, \psi$ is *simultaneously diagonalizable* if there exists a basis $\mathcal{B}$ of $V$ consisting of eigenvectors for both $\varphi$ and $\psi$. Notice that if $\varphi, \psi$ are simultaneously diagonalizable, then they commute because they commute on the basis of eigenvectors. We will show that the converse is also true.

**Exercise 1.1.** *If $\varphi, \psi$ are diagonalizable operators on a finite dimensional vector space $V$, then they are simultaneously diagonalizable if and only if they commute.*

Let's start the proof. Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of $\varphi$. Since $\varphi$ is diagonalizable, there exists a decomposition of $V$ into eigenspaces $V_{\lambda_i} = \{v \in V : \varphi(v) = \lambda_i v\}$,

$$V = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_m}. \tag{1}$$

Note that for an eigenspace $V_\lambda$ of $\varphi$, if we take an eigenvector $v \in V_\lambda$, then $\varphi(\psi(v)) = \psi(\varphi(v)) = \psi(\lambda v) = \lambda\psi(v)$. This means that $\psi(V_\lambda) \subseteq V_\lambda$, so it makes sense to restrict the operator $\psi$ to the space $V_\lambda$. Once we show that $\psi|_{V_{\lambda_i}}$ is diagonalizable for every eigenspace $V_{\lambda_i}$, we may take a basis $\mathcal{B}_i$ of eigenvectors for $\psi$ in $V_{\lambda_i}$. Since the vectors in $\mathcal{B}_i$ belong to $V_{\lambda_i}$, they are eigenvectors for $\varphi$ too. Then the basis $\mathcal{B} = \bigcup_i \mathcal{B}_i$ simultaneously diagonalizes $\varphi$ and $\psi$. Therefore, it suffices to prove the following statement.

**Lemma 1.2.** *Let $W$ be a subspace of a finite dimensional vector space $V$. If $\varphi$ is a linear operator on $V$ such that $\varphi$ is diagonalizable and $\varphi(W) \subseteq W$, then $\varphi|_W$ is diagonalizable.*

We provide two proofs of Lemma 1.2. One argument is elementary, in the sense that it uses no further definitions than the ones given above, but there is a clever induction argument. The other uses the minimal polynomial. Then we give some extensions of Exercise 1.1 and Lemma 1.2.

## 2. Elementary proof of Lemma 1.2

As above, let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of $\varphi$, and decompose $V$ into the eigenspaces of $\varphi$ as in equation (1). It would be nice if we can choose an basis of eigenvectors for $W$ from the decomposition of $V$ in (1) because the bases of $W \cap V_{\lambda_i}$ would be eigenvectors of $\varphi|_W$ and we obtain a basis for $W$ by concatenating the bases of

$W \cap V_{\lambda_i}$. It turns out that this is true. Specifically, we will show that we can decompose $W$ into a direct sum of its factors

$$W = (W \cap V_{\lambda_1}) \oplus (W \cap V_{\lambda_2}) \oplus \cdots \oplus (W \cap V_{\lambda_m}). \tag{2}$$

Since $W \cap V_{\lambda_i} \subseteq W$, it suffices to show the inclusion $W \subseteq \bigoplus_i W \cap V_{\lambda_i}$. For that, we show that we can decompose $w = v_1 + v_2 + \cdots + v_m$ for eigenvectors $v_i \in W \cap V_{\lambda_i}$. We prove the following statement by induction on $m$: if $m$ is minimal such that $w = v_1 + \cdots + v_m$ *with* $v_i \neq 0$, then $v_i \in W \cap V_{\lambda_i}$. Note that since $\varphi$ is diagonalizable, any vector can be decomposed into a sum of eigenvectors, so such an $m$ must exist.

If $w = v_i$ where $v_i \in V_{\lambda_i}$, then $w \in W$ implies $v_i \in W \cap V_{\lambda_i}$. Suppose $w = v_1 + \cdots + v_k \in W$, where $v_i \in V_{\lambda_i}$ and are nonzero. Then

$$\varphi(w) - \lambda_1 w = (\lambda_2 - \lambda_1)v_2 + (\lambda_3 - \lambda_1)v_3 + \cdots + (\lambda_k - \lambda_1)v_k. \tag{3}$$

Since $\varphi(w) \in W$, we know that $\varphi(w) - \lambda_1 w \in W$. Thus the right side is a vector in $W$ which is a sum of $k-1$ eigenvectors the inductive hypothesis on (3), $v_2, \ldots, v_k \in W$. Also, $v_1 = w - (v_2 + \cdots + v_k) \in W$. This completes the induction.

*Remark.* When considering the decomposition of any $w = v_1 + \cdots + v_m \in W$, we can automatically show $v_i \in W$ for every $i = 1, \ldots, m$ by considering the operator

$$\pi_i = \prod_{j \neq i} \frac{\varphi - \lambda_j}{\lambda_j - \lambda_i},$$

where product in this case means composition of operators. The operator is a polynomial in $\varphi$, so $\pi_i(W) \subseteq W$. Notice that $\pi_i(v_i) = v_i$ and $\pi_i(v_j) = 0$ for all $j \neq i$. So, $\pi_i(w) = \sum_{j=1}^{m} \pi_i(v_j) = v_i$, implying $v_i \in W$.

## 3. Proof of Lemma 1.2 via the minimal polynomial

The *minimal polynomial* of an operator $\varphi : V \to V$ is a polynomial $m_\varphi(x) \in k[x]$ such that $m_\varphi$ is monic and it has least degree for which $m_\varphi(\varphi) = 0$. We will use the basic properties of the minimal polynomial, see for example Keith Conrad's notes [1].

**Theorem 3.1.** *Let $\varphi : V \to V$ be an operator on $V$. A polynomial $p(x) \in k[x]$ satisfies $p(\varphi) = 0$ if and only if $m_\varphi(x) \mid p(x)$.*

**Theorem 3.2.** *Let $\varphi : V \to V$ be an operator on $V$. Then $\varphi$ is diagonalizable if and only if $m_\varphi$ can be written as a product of linear factors in $k[x]$ and $m_\varphi$ has distinct roots.*

Using these properties, we can give a shorter proof of Lemma 1.2.

*Proof.* Let $\phi = \varphi|_W$. From Theorem 3.1, $m_\varphi(\varphi) = 0$, which means the restriction of $m_\varphi$ to $W$ is 0. Then

$$m_\varphi(\phi) = m_\varphi(\varphi|_W) = m_\varphi(\varphi)|_W = 0.$$

So, $m_\phi \mid m_\varphi$. Since $\varphi$ is diagonalizable, by Theorem 3.2, $m_\varphi$ splits into a product of linear factors and has distinct roots. This means that the minimal polynomial of $\phi$ has no repeated factors and splits. Therefore, $\phi$ is diagonalizable. □

## 4. Extensions

The following is an extension of Exercise 1.1.

**Exercise 4.1.** *Let $\mathcal{F} = \{\varphi_i\}_{i \in I}$ be a collection of commuting linear operators on a finite dimensional vector space $V$. If each $\varphi_i$ is diagonalizable on $V$, then the operators in $\mathcal{F}$ are simultaneously diagonalizable.*

See [1] for a proof that follows two steps: prove the statement for a finite number of operators inductively, and then prove the general statement by finding a basis for the subspace spanned by $\{\varphi_i\}_{i \in I}$ inside $\mathrm{Hom}(V, V)$.

There is also an extension of Lemma 1.2 if we consider the induced mapping on the quotient space $V/W$, defined by $\overline{\varphi} : v + W \mapsto \varphi(v) + W$.

**Lemma 4.2.** *Let $W$ be a subspace of a finite dimensional vector space $V$. If $\varphi$ is a linear operator on $V$ such that $\varphi$ is diagonalizable and $\varphi(W) \subseteq W$, then $\varphi|_W$ and $\overline{\varphi}$ are diagonalizable.*

## References

[1] Keith Conrad. The minimal polynomial and some applications. Online notes.