CAN YOU HEAR THE SHAPE OF A DRUM?

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Abstract. We survey the problem of constructing isospectral manifolds.

1. Introduction

Suppose we are given a domain $\Omega \subseteq \mathbb{R}^2$ with smooth boundary Γ. Think of Ω as a membrane, and suppose we set it in motion with its boundary Γ fixed. Let $F(x, y; t) = F(\rho; t)$ be the vertical displacement, perpendicular to the plane containing Ω, of the particle $\rho = (x, y)$ at time t. It is a theorem from classical mechanics that F satisfies the *wave equation*

$$\frac{\partial^2 F}{\partial t^2} = c^2 \nabla^2 F,$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the *spatial Laplacian operator*. Special solutions to the above equation are *pure tones* $F(\rho; t) = U(\rho)e^{i\omega t}$. Pure tones $U(\rho)e^{i\omega t}$ satisfy the wave equation if and only if

$$c^2\nabla^2 U + \omega^2 U = 0 \text{ and } U = 0 \text{ on } \Gamma.$$

By "$U = 0$ on Γ" we mean that $U(\rho) \to 0$ as ρ approaches a point of Γ. Showing that are a discrete set of frequencies ω satisfying the above equation was an important problem in 19th century mathematical physics: Poincaré and many others struggled with it. In the beginning of the 20th century, it was shown that the Laplacian does have a discrete spectrum.

Kac considers the membrane Ω to be a "drum" which is set in motion. He attributes hearing the problem from Salomon Bochner around 10 years before his article [Kac66] was published. The problem "can you hear the shape of a drum?", stripped of all pictoresque language, is the problem of determining the isometry class of Ω given we know all of the eigenvalues λ of the eigenvalue problem

$$\begin{aligned} c^2\nabla^2 U + \lambda U &= 0 \text{ in } \Omega, \\ U &= 0 \text{ on } \Gamma. \end{aligned}$$

One of the main goals of this paper is to answer Kac's question in the negative. Interestingly, even though you cannot "hear" the domain up to isometry, but you can "hear" the area. In 1911, Weyl proved that if $N(\lambda)$ is the number of eigenvalues of Ω less than or equal to λ, then

$$N(\lambda) \sim \frac{|\Omega|}{2\pi}\lambda,$$

which means that $\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda} = \frac{|\Omega|}{2\pi}$. In fact, he proved that for any bounded domain $\Omega \subseteq \mathbb{R}^d$,

$$N(\lambda) \sim (2\pi)^{-d}\omega_d \operatorname{vol}(\Omega)\lambda^{d/2},$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$.

In this paper, we survey the history and important results related to isospectral domains, and more generally, isospectral manifolds.

1.1. **Organization of the paper.** We go over relevant background in Section 2. In Section 3, we prove the existence of two flat 16-dimensional tori which are isospectral and not isometric, a result by Milnor [Mil64] with a short and elegant proof. In Section 4, we motivate and outline the proof of a celebrated construction method of isospectral manifolds by Sunada [Sun85]. Finally, in Section 5, we provide a construction method of isospectral non-isometric planar domains by Buser, Conway, Doyle, and Semmler [BCDS94], settling Kac's question in the negative.

2. Background

Let (M, g) be a compact Riemannian manifold with boundary. Then M has a Laplace operator Δ_M that acts on smooth functions f on M, defined by $\Delta_M(f) = -\operatorname{div}(\operatorname{grad} f)$[1]. It can be shown using the spectral theorem of compact self-adjoint operators, that the eigenspaces of Δ_M are finite dimensional and that the Dirichlet eigenvalues are real, positive, and have no limit point. Thus, they can be arranged in increasing order

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots, \ \lambda_n \to \infty.$$

There moreover exists an orthonormal sequence of $C^\infty(M)$ functions $\{1 = f_0, f_1, f_2, \ldots\}$ with $\Delta_M f_i = \lambda_i f_i$ that contains a basis for every eigenspace and generates a dense subspace of $L^2(M)$ in the L^2-norm topology. If M has boundary, we can ensure that the f_i are vanish on the boundary of M. We then call the eigenvalues in the spectrum *Dirichlet eigenvalues* and the eigenfunctions *Dirichlet eigenfunctions*.

The ordered sequence of nonzero eigenvalues of Δ_M, listed with multipicity, is called the *eigenvalue spectrum* of M, denoted $\lambda(M)$. Two Riemannian manifolds are *isospectral* if their spectra coincide, counted with multiplicities.

Definition 2.1. Let M an n-dimensional compact connected Riemannian with eigenvalue spectrum $\lambda(M) = (\lambda_i)_{i\geq 1}$. The (Minakshisundaram-Pleijel) *zeta function* of M is defined by the generalized Dirichlet series

$$\zeta_M(s) = \sum_{i\geq 1} \lambda_i^{-s}.$$

In a celebrated article of Minakshisundaram and Pleijel [MP49], they used the above zeta function to prove Weyl's law. They also showed that $\zeta_M(s)$ converges absolutely and uniformly to a holomorphic function on some right half-plane and has a meromorphic continuation to $\mathbb{C}$ that is holomorphic at simple poles at integers $1, \ldots, n/2$ if n is even and half-integers $n/2, n/2 - 1, \ldots$ if n is odd.

Proposition 2.2. *M_1 and M_2 are isospectral if and only if $\zeta_{M_1}(s) = \zeta_{M_2}(s)$.*

Proof. This proof follows Andrew Sutherland's lecture notes on arithmetic eqivalence and isospectrality. The forward direction is immediate. Suppose $\zeta_{M_1}(s) = \zeta_{M_2}(s)$ but the eigenvalue spectra $\lambda(M_1)$ and $\lambda(M_2)$ do not coincide. Without loss of generality, we can choose j such that $\lambda_j(M_1) > \lambda_j(M_2)$ and $\lambda_i(M_1) = \lambda_i(M_2)$ for all $1 \leq i < j$. Note that this implies that $\lambda_j(M_1) > \lambda_i(M_2)$ for all $i \geq j$. Let n_j be the multiplicity of $\lambda_j(M_1)$ in $\lambda(M_1)$. Since the zeta functions ζ_{M_i} converge absolutely and uniformly on some right half-plane, we can find

[1] The sign convention is to guarantee that the eigenvalues increase.

$\sigma \in \mathbb{R}$ such that $\zeta_{M_1}(s)$ and $\zeta_{M_2}(s)$ converge absolutely and uniformly for all $s \geq \sigma$. This implies that for real $t \geq \sigma$,

$$\zeta_{M_1}(t) - \zeta_{M_2}(t) \sim n_j \lambda_j(M_1)^{-t}.$$

This contradicts $\zeta_{M_1}(s) = \zeta_{M_2}(s)$, which concludes the proof. □

3. 16-dimensional tori

A *flat torus* is a Riemannian manifold of the form $\mathbb{R}^n/L$, where L is a lattice of rank n. The torus is called flat because it has zero Gaussian curvature everywhere. Using this fact, Milnor [Mil64] proved the following:

Theorem 3.1. *There exist a pair of* 16*-dimensional flat tori which are isospectral and non-isometric.*

Proof. For a lattice L of $\mathbb{R}^n$, each $y \in L^*$ determines an eigenfunction $f(x) = \exp(2\pi i x \cdot y)$ for the Laplace operator on $\mathbb{R}^n/L$. The corresponnding eigenvalue λ is $(2\pi)^2 y \cdot y$. So, the number of eigenvalues less than or equal to $(2\pi r)^2$ is equal to the number of points of L^* lying within a ball of radius r about the origin.

Define the *dual* L^* of a lattice L to be all $y \in \mathbb{R}^n$ such that $x \cdot y$ is an integer for all $x \in L$, and L is *self-dual* if $L = L^*$. According to Witt [Wit41], there is a pair of self-dual lattices $L_1, L_2 \in \mathbb{R}^{16}$ such that there is no rotation of $\mathbb{R}^{16}$ carrying L_1 to L_2 such that each ball B around the origin contains the same number of points of L_1 as L_2. Self-dual lattices have the same number of points inside a ball of radius r because they have determinant of norm 1. It follows that $\mathbb{R}^{16}/L_1$ and $\mathbb{R}^{16}/L_2$ are not isometric, but do have the same sequence of eigenvalues. □

After Milnor's example, many isospectral pairs of manifolds in dimension two and higher were constructed. Vignéras [Vig80] constructed examples for Riemann surfaces with constant negative curvature, and Ikeda [Ike83] constructed isospectral lens spaces with constant curvature.

4. Sunada's theorem

Sunada's theorem has its origins in number theory. Given an algebraic number field K (a finite extension of $\mathbb{Q}$), each prime $p \in \mathbb{N}$ has a prime ideal decomposition with respect to K, and we can package the degrees of the prime ideals factors into an increasing sequence $\ell[K](p)$ called the *length of p with respect to K*. We call two number fields K_1 and K_2 *isospectral* if for all large enough primes p, $\ell[K_1](p) = \ell[K_2](p)$. In 1880, Kronecker [Kro80] asked whether two number fields are isomorphic if they have the same spectrum.

In 1925, building on the work on Hurwitz, Bauer, and others, Gassmann [Gas26] showed the existence of two isospectral non-isomorphic number fields is equivalent to displaying a pair of *finite* groups with a particular property, called *Gassmann equivalent* or *almost-conjugate*, see Definition 4.1. He then constructed a pair with this property, giving a counter example to Kronecker's conjecture. In modern language, Gassmann's result is stated in terms of *the zeta function associated to a number field* because it is known that is an equivalent condition for isospectrality of number fields.

See Buser's book [Bus10] for an excellent account on Sunada's theorem.

Proposition. *Let K be a finite Galois extension of $\mathbb{Q}$ with Galois group $G = G(K/\mathbb{Q})$, and let k_1 and k_2 be subfields of K corresponding to subgroups H_1 and H_2 of G respectively. Then the following two conditions are equivalent*

(1) The groups H_1 and H_2 are Gassmann equivalent.
(2) The zeta functions of k_1 and k_2 are the same.

In 1984, Sunada discovered that Buser's argument can be adapted to Riemannian geometry. To state Sunada's theorem, we associate to a pair of finite groups a pair of isospectral manifolds. The groups act by isometries on a given Riemannian manifold, and the examples are the quotients. Denote # by the *cardinality* of a set, and for any group G, denote the conjugacy class of an element $g \in G$ by $[g]$,

$$[g] = \{\sigma g \sigma^{-1} \mid \sigma \in G\}.$$

Definition 4.1. Let G be a finite group. Two subgroups H_1, H_2 of G are called *almost conjuate* or *Gassmann equivalent* if for all $g \in G$,

$$\#([g] \cap H_1) = \#([g] \cap H_2).$$

Observe that conjugate subgroups are almost conjugate.

Theorem 4.2 (Sunada's Theorem)**.** *Let $\pi : M \to M_0$ be a normal finite Riemannian covering with covering transformation group G, and for $i = 1, 2$, let $\pi_i : M_i \to M_0$ be coverings corresponding to the subgroups H_i. If H_1 and H_2 are almost conjugate, then M_1 and M_2 are isospectral.*

4.1. **Proof of Sunada's Theorem.** In this section, we outline the proof of Sunada's theorem. First we develop a trace-formula. See Appendix B for relevant definitions.

Let V be a Hilbert space on which a finite group acts as unitary transformations, and let $A : V \to V$ be a self-adjoint operator of trace class. We assume L commutes with the G-action.

For a subgroup H in G, denote V^H to be the subspace of H-invariant vectors:

$$V^H = \{v \in V \mid hv = v \text{ for any } h \in H\}.$$

In the following trace formula, $\mathrm{tr}(A|V^H)$ with respect to the vector space V^H, and $\mathrm{tr}(hA)$ and $\mathrm{tr}(gA)$ are with respect to the vector space V^H. Let $[G]$ be the set of conjugacy classes of G. Denote the conjugacy class of h in H by $[h]_H$.

Lemma 4.3 (Elementary trace formula)**.** *The restriction of L to V^H is also of trace class, and*

$$\mathrm{tr}(A|V^H) = \frac{1}{\#H} \sum_{[h]_H \subset [H]} \#[h]_H \,\mathrm{tr}(hA) = \frac{1}{\#H} \sum_{[g] \in [G]} \#([g] \cap H)\,\mathrm{tr}(gA).$$

Proof. Consider the projection map P given by

$$P(v) = \frac{1}{\#H} \sum_{h \in H} hv.$$

Then P maps V onto V^H and acts as the identity map on V^H. We also see that $\mathrm{tr}(PA) = \mathrm{tr}(A|V^H)$ by extending a basis of V^H to a basis of V. This implies that

$$\mathrm{tr}(A|V^H) = \frac{1}{\#H} \sum_{h \in H} \mathrm{tr}(hA).$$

Note that trace is invariant under conjugation, so

$$\operatorname{tr}(A|V^H) = \frac{1}{\#H} \sum_{[h]_H \in [H]} \#[h]_H \operatorname{tr}(hA),$$

which gives us the first formula. To obtain the second, we note that for every conjugacy class $[h]_H$ of H and any conjugacy class $[g]$ of G, either $[h]_H \subset [g]$ or $[h]_H \cap [g] = \emptyset$. This implies that we can rewrite the sums as $\sum_{[h]_H \in [h]} = \sum_{[g]\in[G]} \sum_{[h]_H \in [g]\cap H}$. Again by invariance of trace under conjugation, $\operatorname{tr}(hA) = \operatorname{tr}(gA)$ for all h such that $[h]_H \subset [g] \cap H$. This implies that

$$\operatorname{tr}(A|V^H) = \sum_{[h]_H \in [H]} \#[h]_H \operatorname{tr}(hA) = \frac{1}{\#H} \sum_{[g]\in[G]} \sum_{[h]_H \in [g]\cap H} \#[h]_H \operatorname{tr}(gA).$$

From the fact that $\sum_{[h]_H \in [g]\cap H} \#[h]_H = \#([g] \cap H)$, we obtain the second formula. □

Lemma 4.3 implies following corollary.

Corollary 4.4. *Suppose that H_1, H_2 are almost conjugate subgroups of G. Then* $\operatorname{tr}(A|V^{H_1}) = \operatorname{tr}(A|V^{H_2})$.

Let $\pi : M \to M_0$ be a normal finite Riemannian covering with covering transformation group G, and for $i = 1, 2$, let $\pi_i : M_i \to M_0$ be coverings corresponding to the subgroups H_i. See Appendix A for covering space definitions. The crux of Sunada's theorem can now be proved.

Theorem 4.5. *If H_1 and H_2 are almost conjugate, then M/H_1 and M/H_2 are isospectral.*

Proof outline. By Proposition 2.2, it suffices to show that $\zeta_{M_1}(s)$ and $\zeta_{M_2}(s)$ are identical.

To a compact connected Riemannian manifold M, we associate the subset $V_M \subseteq L^2(M)$ and the operator A_M on V_M, defined by

$$\begin{aligned} V_M &= \left\{ f \in L^2(M) : \int_M f\,dx = 0 \right\}, \\ A_M &= (\Delta_M|V)^{-s}. \end{aligned}$$

It is known A_M is of trace class and $\operatorname{tr} A_M = \zeta_M(s)$. From the covering projection map $\widetilde{\omega}_i : M \to M/H_i$, consider the map

$$\begin{aligned} L^2(M_i) &\to L^2(M) \\ f &\mapsto (\#H_i)^{-1/2} f \circ \widetilde{\omega}_i. \end{aligned}$$

This map induces a map of Hilbert spaces

$$\phi_i : V_{M_i} \to V_M^{H_i},$$

where $V_M^{H_i}$ are the H_i-invariant vectors in V_M. It is not difficult to verify that ϕ_i is an isometry. Moreover, by considering the action of ϕ_i on each eigenspace and using linearity of the Laplacian, we see that

$$\phi_i \circ A_{M_i} = A_M \circ \phi_i. \tag{1}$$

Let $(e_k)_k$ be an orthonormal basis for V_{M_i}. Since ϕ_i is an isometry, $(\phi_i(e_k))_k$ is an orthonormal basis for $V_M^{H_i}$. Then

$$\begin{aligned}
\operatorname{tr} A_{M_i} &= \sum_k \langle A_{M_i} e_k, e_k \rangle \\
&= \sum_k \langle \phi_i(A_{M_i} e_k), \phi_i(e_k) \rangle, && (\phi_i \text{ an isometry}) \\
&= \sum_k \langle A_M \phi_i(e_k), \phi_i(e_k) \rangle && (\text{By } (1)) \\
&= \operatorname{tr}(A_M | V_M^{H_i}).
\end{aligned}$$

This implies $\zeta_{M_i}(s) = \operatorname{tr}(A_{M_i}) = \operatorname{tr}(A_M | V_M^{H_i})$. By Corollary 4.4,

$$\zeta_{M_1}(s) = \operatorname{tr}(A_M | V_M^{H_1}) = \operatorname{tr}(A_M | V_M^{H_2}) = \zeta_{M_2}(s).$$

This concludes the proof. □

5. Planar domains

In this section we answer Kac's original question in the negative.

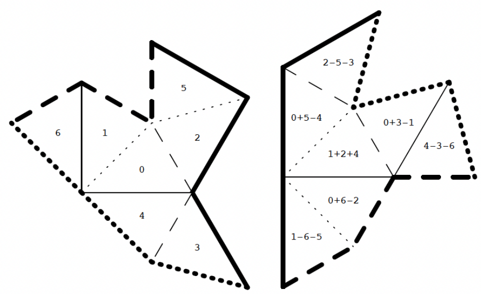

Figure 1. Warped Propeller, from [BCDS94, Figure 3]

Theorem 5.1. *The two domains given in Figure 1 are Dirichlet isospectral and not isometric.*

Proof. We consider two arrangements of seven equilateral triangles, which the paper calls 'propellers', and replace each equilateral triangle with a scalene triangle such that any two triangles sharing an edge are reflections of one another across the edge. These 'warped propellers', label the left one L and the right one by R are shown in Figure 1.

First we show that warped propellers L and R are not isometric. This comes directly from the construction: the only isometry carrying L to R must carry the central triangle of L to the central triangle of R. Since the triangles are scalene, the only such isometry is a translation, but a translation does not carry the rest of L to R. This shows that L and R are not isometric.

To show that warped propellers L and R are Dirichlet isospectral, we construct two maps from the λ-eigenspace of L to the λ-eigenspace of R. They use a technique from Riemann surfaces called *transplantation* first developed by Buser [Bus86]. In the case of planar domains, the technique is particularly easy.

Let f_L be a Dirichlet eigenfunction of L with eigenvalue λ. Using f_L, they construct a Dirichlet eigenfunction f_R of R with eigenvalue λ, called the *transplantation* of f_L. Label the triangles of L by $1, \ldots, 7$, and for each $i = 1, \ldots, 7$, let $f_{L,i}$ be the restriction of f_L to triangle i. For convenience, let $\mathbf{1}, \ldots, \mathbf{7}$ be the eigenfunctions $f_{L,1}, \ldots, f_{L,7}$. In the central triangle, we put the function $f_{L,1} \circ \tau_1 + f_{L,2} \circ \tau_2 + f_{L,4} \circ \tau_4$, where τ_k is the isometry from the central triangle of propeller R to the triangle i in propeller L. For convenience, label the function $f_{L,1} \circ \tau_1 + f_{L,2} \circ \tau_2 + f_{L,4} \circ \tau_4$ with $\mathbf{1} + \mathbf{2} + \mathbf{4}$. In the same way, construct functions on the other triangle in propeller R using the numbers shown.

We claim that the function f_R obtained from pasting together all the functions is well-defined. Since f_L is a Dirichlet eigenfunction, it vanishes on the boundary of L, or equivalently by the reflection principle, extends to the function $-f_L$ in a neighborhood of the boundary of L. This means that the functions $\mathbf{1}, \mathbf{2}, \mathbf{4}$ extend across their respective boundary segments to the functions $\mathbf{0}, \mathbf{5}, -\mathbf{4}$, so the sum $\mathbf{1}+\mathbf{2}+\mathbf{4}$ extends to the function $\mathbf{0}+\mathbf{5}-\mathbf{4}$ across the shared boundary segment. Using the same technique, we can show that the other pairs of functions defined on the triangles of R paste together properly. Therefore, f_R is well-defined.

Such a map from the λ-eigenspace of L to the λ-eigenspace of R is checked to be nonsingular, so the dimension of the λ-eigenspace of L is at most the the dimension of the λ-eigenspace of R. The same transplantation technique works to define a map from the λ-eigenspace of R to the λ-eigenspace of L. This shows that the λ-eigenspaces of L and R have the same dimension. Therefore, the domains L and R are Neumann isospectral. □

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Appendix A. Covering maps

- Let $(\widetilde{M}, h)$ and (M, g) be Riemannian manifolds. $\pi : \widetilde{M} \to M$ is a *Riemannian covering map* if π is a smooth covering map and π is a local isometry.
- A *covering (deck) transformation* of π is a diffeomorphism $\varphi : \widetilde{M} \to \widetilde{M}$ such that $\pi \circ \varphi = \pi$, i.e. the following diagram commutes:

$$\begin{tikzcd} \widetilde{M} \arrow[rr, "\varphi"] \arrow[dr, "\pi"] & & \widetilde{M} \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

This set of covering transformations forms the *covering transformation group* $\mathrm{Deck}(\pi)$. $\mathrm{Deck}(\pi)$ acts on E. A covering map is *normal* if $\mathrm{Deck}(\pi)/\widetilde{M} \cong M$, i.e. for all $x \in M$ and any $y_0, y_1 \in \pi^{-1}(x)$, there exists $\varphi \in \mathrm{Deck}(\pi)$ such that $\pi(y_0) = y_1$. If the fibers π are finite, then the covering map is called *finite*.

Appendix B. Trace of an operator

- A *Hilbert space* H is a real or complex inner product space that is also a complete metric space with respect to the distance function induced by the inner product.
- Let $A : H \to H$ be a bounded linear operator on H, which is self-adjoint and positive semi-definite. A is *unitary* if it is a bijection and preserves the inner product. Define the *trace* of A to be

$$\mathrm{tr}(A) = \sum_k \langle Ae_k, e_k \rangle$$

for an orthonormal basis (e_k) of H. It can be shown that it does not matter which orthonormal basis (e_k) one chooses, so the trace is well-defined. The operator A is of *trace class* if $\mathrm{tr}(A)$ is finite.